Differential Geometry as a Paradigm for MDO

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Definition of MDO

- Optimization of coupled subsystems
- Special mathematical structure

$$\min f(\mathbf{x}, \mathbf{y}, \mathbf{z})$$

$$\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq \mathbf{0}$$

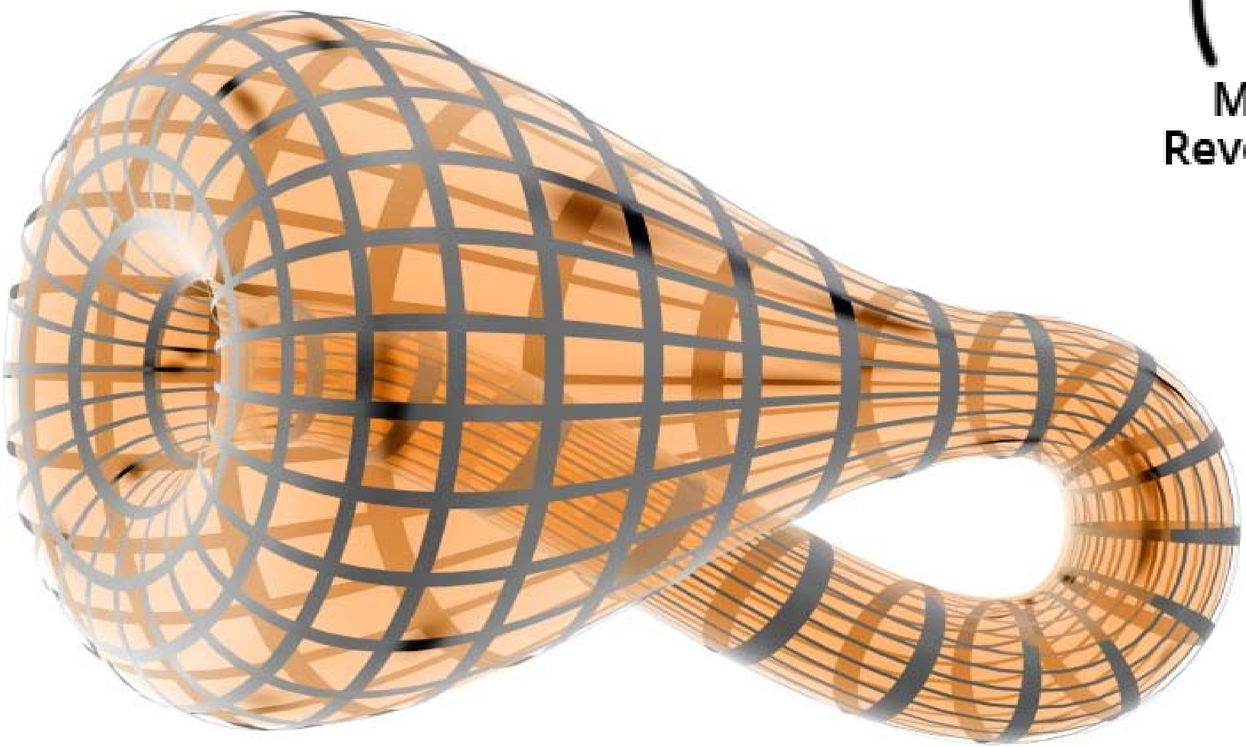
$$\mathbf{h}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{0}$$

$$y^i = \psi^i(\mathbf{x}^i, \tilde{\mathbf{y}}^i, \mathbf{z}), \;\; i = 1, 2, \ldots$$

$$\mathbf{w} = \left\{ \begin{array}{c} \mathbf{x} \\ \mathbf{z} \end{array} \right\} \qquad \mathbf{v} = \left\{ \begin{array}{c} \mathbf{w} \\ \mathbf{y} \end{array} \right\}$$

Why a Differential Geometry Framework?

- Three research foci
- Lack of consensus
- Kuhn's philosophy of science
- Differential geometry



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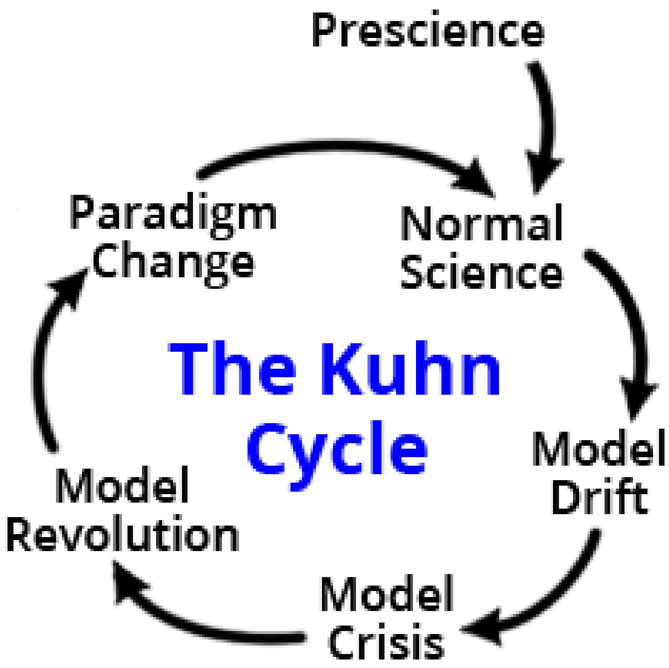
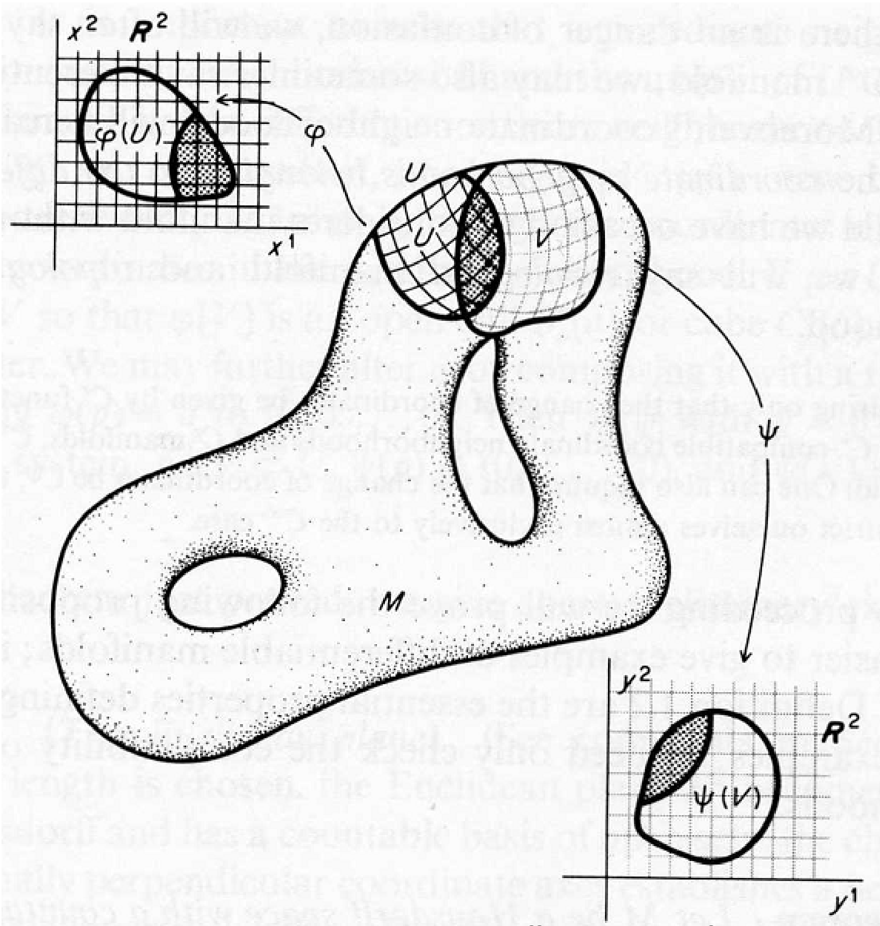


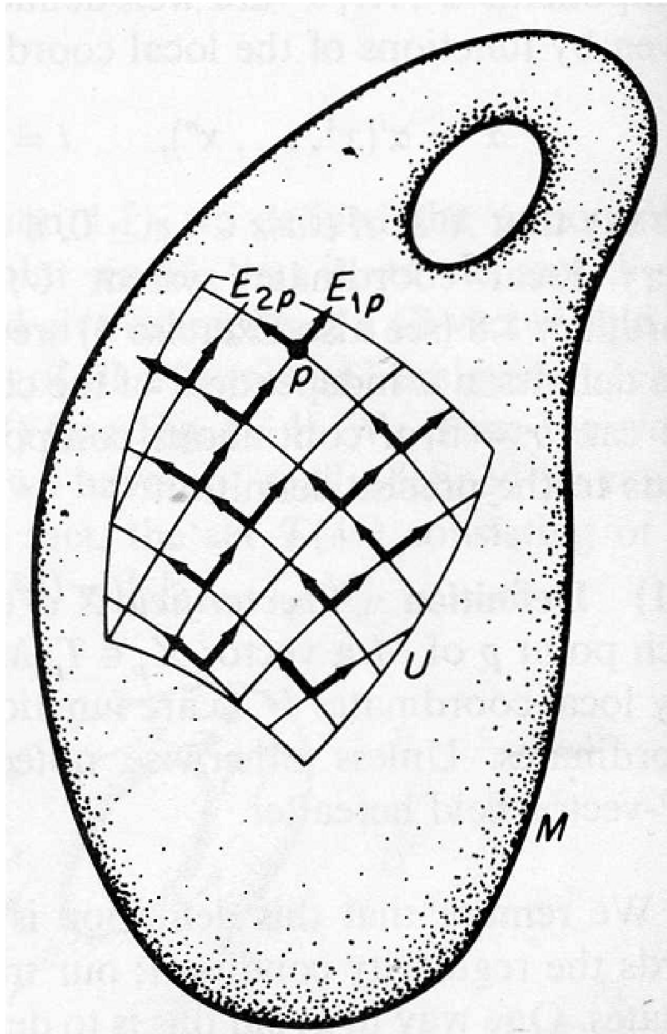
Twink.org: KuhnCycle_BasicCycle.png. URL http://www.thwink.org/sustain/glossary/KuhnCycle.htm

Differential Geometry

- Manifolds and charts
- Tangent spaces and basis vectors



Boothby, W.M.: An Introduction to Differentiable Manifolds and Riemannian Geometry. Academic Press, Inc., Boston (1983)



Boothby, W.M.: An Introduction to Differentiable Manifolds and Riemannian Geometry. Academic Press, Inc., Boston (1983)

Riemannian Geometry

- The metric tensor
- Other geometric quantities
- Covariant derivatives

$$g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j \qquad \langle \mathbf{u} \cdot \mathbf{v} \rangle = g_{ij} u^i v^j \qquad ds^2 = g_{ij} dx^i dx^j$$

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial w^l} + \frac{\partial g_{ml}}{\partial w^k} - \frac{\partial g_{kl}}{\partial w^m} \right)$$

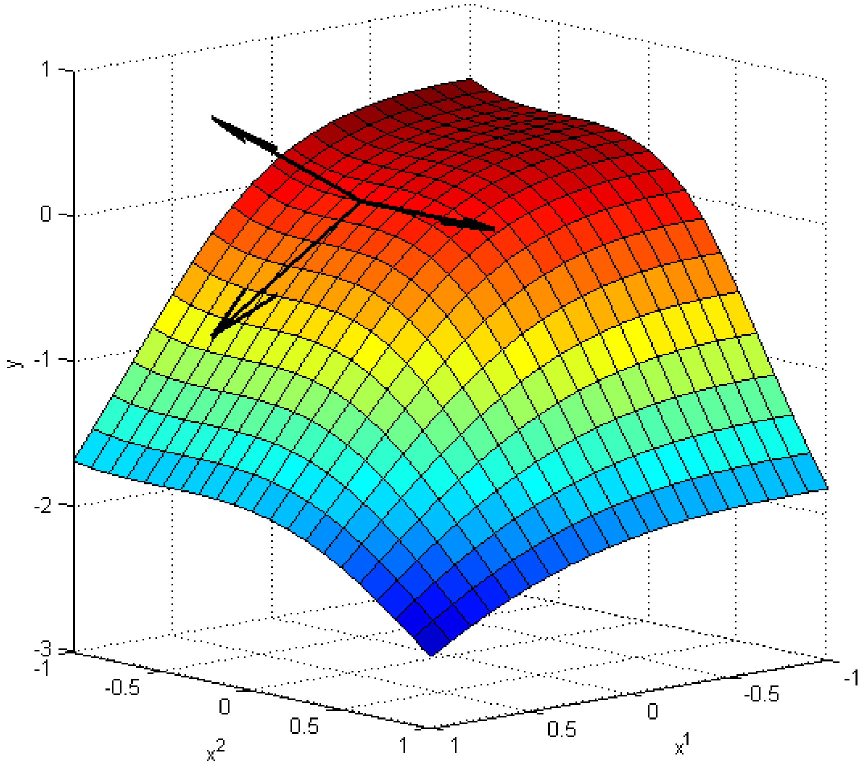
$$R^i_{jkl} = \frac{\partial}{\partial w^k} \left(\Gamma^i_{jl} \right) - \frac{\partial}{\partial w^l} \left(\Gamma^i_{jk} \right) + \Gamma^m_{jl} \Gamma^i_{mk} - \Gamma^m_{jk} \Gamma^i_{ml}$$

$$\eta_{;i} = \eta_{,i} = \frac{\partial \eta}{\partial w^i} \qquad \eta_{;ij} = \eta_{,ij} - \Gamma^l_{ij} \eta_{,l}$$

Translating MDO

- Tangent vectors
- The metric tensor

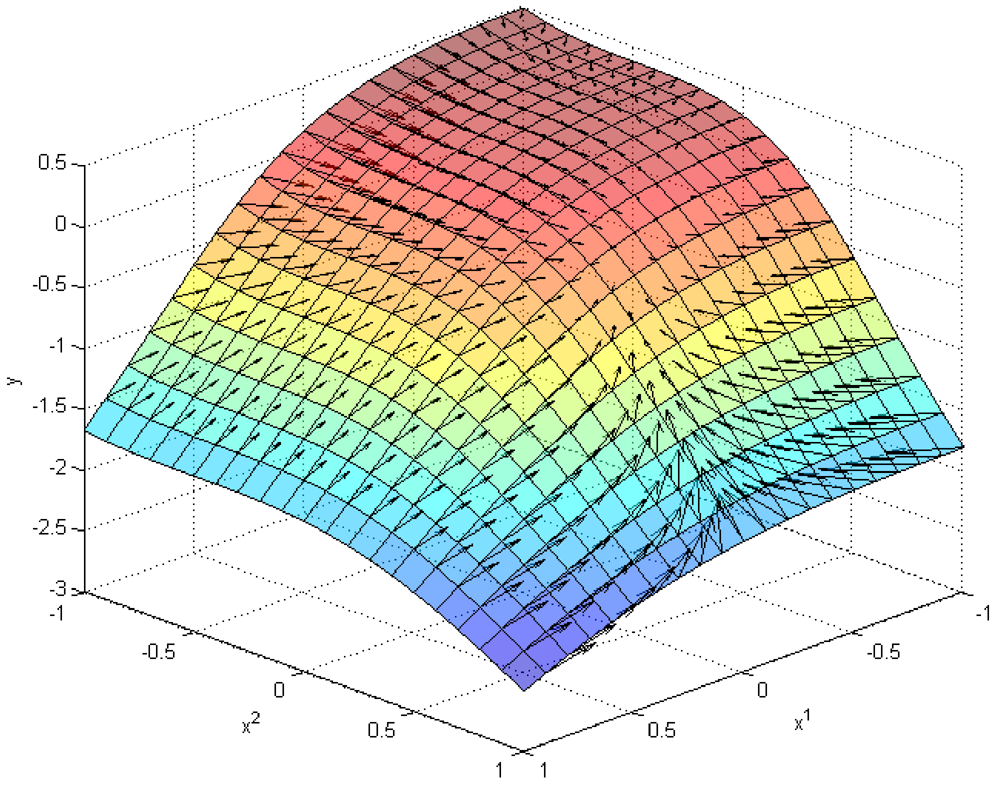
$$\mathbf{t}_i = \frac{\partial \mathbf{v}}{\partial w^i} = \left\{ \begin{array}{c} \frac{\partial v^1}{\partial w^i} \\ \vdots \\ \frac{\partial v^m}{\partial w^i} \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \frac{\partial y^1}{\partial w^i} \\ \vdots \\ \frac{\partial y^{m-n}}{\partial w^i} \end{array} \right\}, \; i = 1, 2, \ldots, n$$

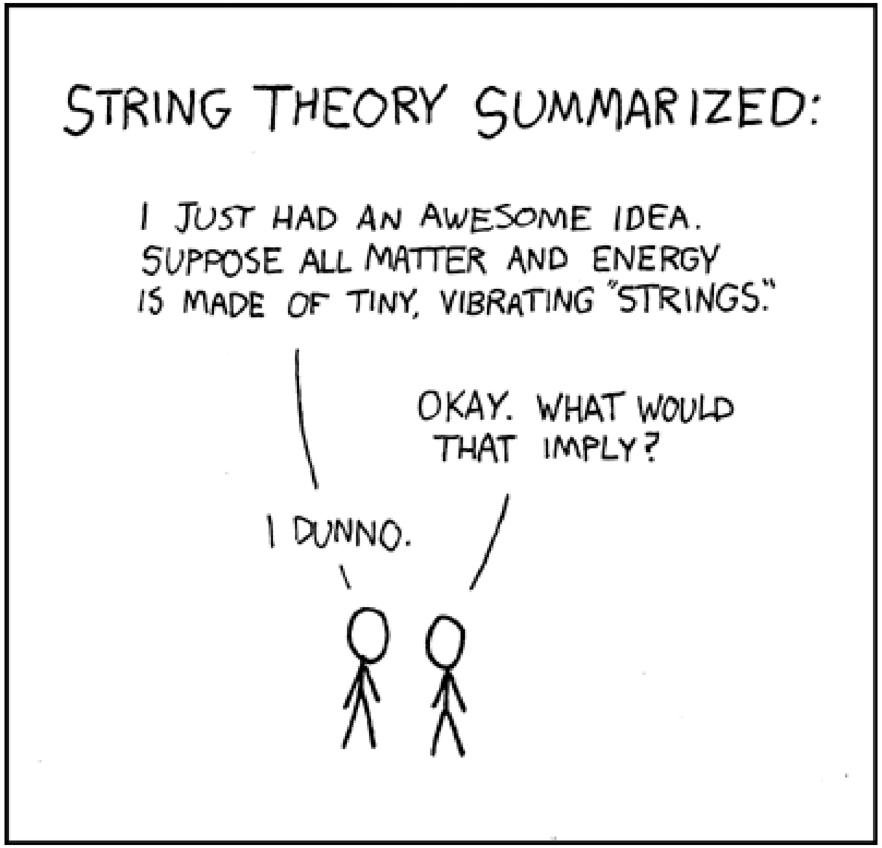


$$g_{ij} = \frac{\partial v^k}{\partial w^i}\frac{\partial v^k}{\partial w^j} = \delta_{ij} + \frac{\partial y^k}{\partial w^i}\frac{\partial y^k}{\partial w^j}$$

Areas of Inquiry

- Analogies with dynamics
- Correlations with numerical indicators





XKCD: String theory. URL http://xkcd.com/171/

Design Coupling – Derivation

- Design coupling and disciplinary projections
- Dot products and the metric tensor
- A normalized metric tensor
- Projection bounds

$$g_{ij} = \frac{\partial v^k}{\partial w^i}\frac{\partial v^k}{\partial w^j} = \frac{\partial \mathbf{v}}{\partial w^i} \cdot \frac{\partial \mathbf{v}}{\partial w^j}$$

$$g_{ij} = \begin{bmatrix} X_1 \cdot X_1 & \cdots & X_1 \cdot X_N & X_1 \cdot Z \\ \vdots & \ddots & \vdots & \vdots \\ X_N \cdot X_1 & \cdots & X_N \cdot X_N & X_N \cdot Z \\ Z \cdot X_1 & \cdots & Z \cdot X_N & Z \cdot Z \end{bmatrix}$$

$$\tilde{g}_{ij} = \frac{\frac{\partial \mathbf{v}}{\partial w^i} \cdot \frac{\partial \mathbf{v}}{\partial w^j}}{\left\| \frac{\partial \mathbf{v}}{\partial w^i} \right\| \left\| \frac{\partial \mathbf{v}}{\partial w^j} \right\|} = \frac{g_{ij}}{\sqrt{g_{ii} g_{jj}}}$$

$$\boldsymbol{\delta} = \left[\tilde{X}_i \cdot \tilde{X}_j\right] \boldsymbol{\epsilon}$$

$$[X_i \cdot X_j] = \begin{bmatrix} \frac{\partial \mathbf{v}}{\partial w^{k_1}} \cdot \frac{\partial \mathbf{v}}{\partial w^{l_1}} & \cdots & \frac{\partial \mathbf{v}}{\partial w^{k_1}} \cdot \frac{\partial \mathbf{v}}{\partial w^{l_m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{v}}{\partial w^{k_n}} \cdot \frac{\partial \mathbf{v}}{\partial w^{l_1}} & \cdots & \frac{\partial \mathbf{v}}{\partial w^{k_n}} \cdot \frac{\partial \mathbf{v}}{\partial w^{l_m}} \end{bmatrix}$$

$$\|\boldsymbol{\delta}\| = \left\| \left[\tilde{X}_i \cdot \tilde{X}_j\right] \boldsymbol{\epsilon} \right\| \leq \left\| \left[\tilde{X}_i \cdot \tilde{X}_j\right] \right\| \|\boldsymbol{\epsilon}\|$$

$$\|\boldsymbol{\delta}\| \leq \left\| \left[\tilde{X}_i \cdot \tilde{X}_j\right] \right\|$$

$$[X_i \cdot X_j] = [X_j \cdot X_i]^T$$

Design Coupling – Properties

- Fundamental geometric property
- Bounded coupling values
- Decomposable metric
- Uncoupled disciplines and product manifolds

$$\|\cdot\|_2 \leq \sqrt{\|\cdot\|_1 \|\cdot\|_\infty} \Rightarrow \left\| \left[\tilde{X}_i \cdot \tilde{X}_j \right] \right\|_2 \leq \sqrt{nm}$$

$$[X_i \cdot X_j] = 0$$

$$g_{ij}^{(M \times N)} = \left[\begin{array}{cc} g_{ij}^{(M)} & 0 \\ 0 & g_{ij}^{(N)} \end{array} \right]$$

IDF – Construction

$$\min f(\mathbf{x},\mathbf{y},\mathbf{z})$$

$$\mathbf{g}(\mathbf{x},\mathbf{y},\mathbf{z}) \leq \mathbf{0}$$

$$\mathbf{h}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{0}$$

$$y^i = \psi^i(\mathbf{x}^i, \tilde{\mathbf{y}}^i, \mathbf{z}), \;\; i = 1, 2, \ldots$$

$$\min_{\mathbf{x},\mathbf{z},\mathbf{s}} F = f(\mathbf{x},\mathbf{y},\mathbf{z}) + \phi\left(\mathbf{s} - \mathbf{y}\right)$$

$$\mathbf{g}(\mathbf{x},\mathbf{y},\mathbf{z}) \leq \mathbf{0}$$

$$\mathbf{y} = \boldsymbol{\psi}(\mathbf{x},\mathbf{s},\mathbf{z})$$

$$\mathbf{u} = \left\{ \begin{array}{c} \mathbf{w} \\ \mathbf{s} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{x} \\ \mathbf{z} \\ \mathbf{s} \end{array} \right\}$$

$$\mathbf{t}_i = \frac{\partial\left(\mathbf{u},\mathbf{y}\right)}{\partial w^i} = \left\{ \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \frac{\partial y^1}{\partial u^i} \\ \vdots \\ \frac{\partial y^{m-n}}{\partial u^i} \end{array} \right\}, i = 1, 2, \ldots, m$$

$$g_{ij} = \delta_{ij} + \frac{\partial y^k}{\partial u^i}\frac{\partial y^k}{\partial u^j}$$

IDF – Discussion

- Partial decoupling
- Discipline-dependent performance

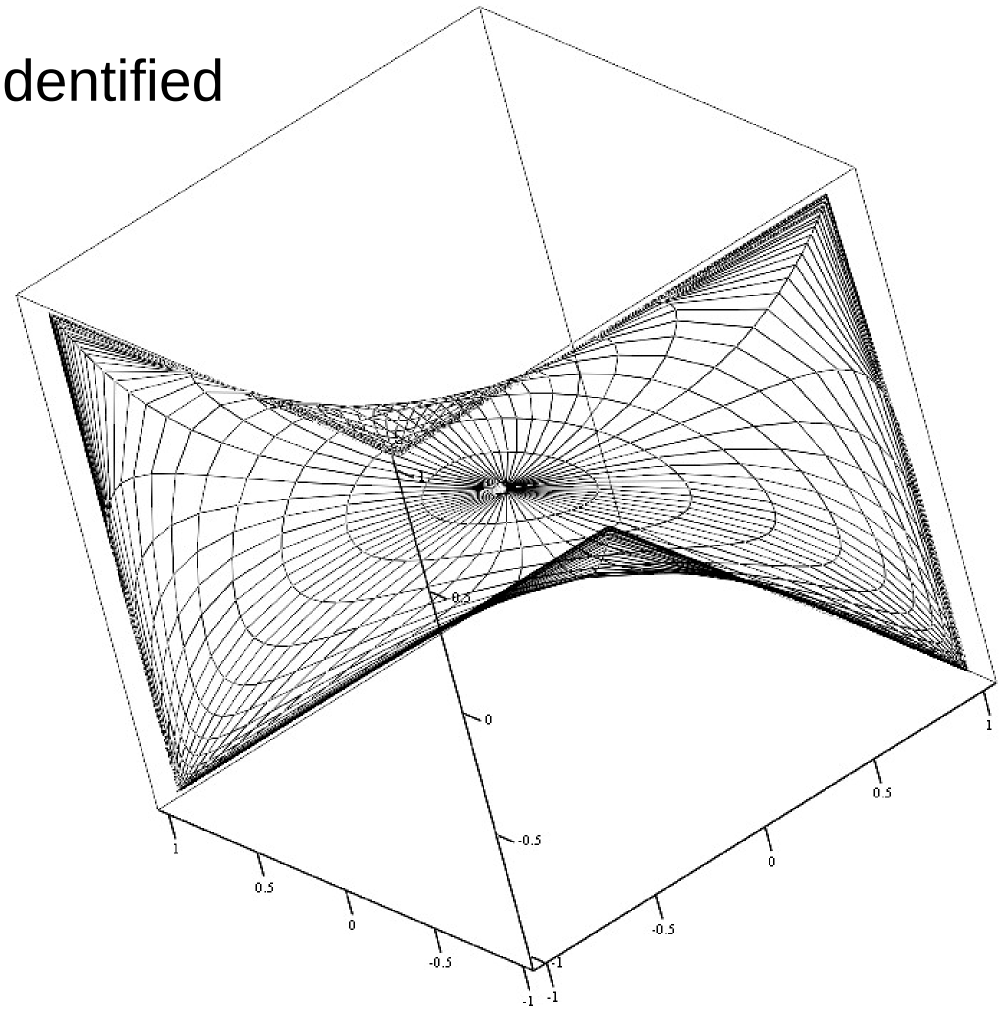
$$\frac{dF}{dw^i} = \frac{\partial f}{\partial w^i} + \frac{\partial f}{\partial y^k}\frac{\partial y^k}{\partial w^i} - \phi_{,k}\frac{\partial y^k}{\partial w^i}$$

$$\frac{d^2F}{dw^i dw^j} = \frac{\partial^2 f}{\partial w^i \partial w^j} + \frac{\partial^2 f}{\partial w^i \partial y^k}\frac{\partial y^k}{\partial w^j} + \frac{\partial^2 f}{\partial y^k \partial w^j}\frac{\partial y^k}{\partial w^i} + \frac{\partial f}{\partial y^k}\frac{\partial^2 y^k}{\partial w^i \partial w^j} + \frac{\partial^2 f}{\partial y^k \partial y^l}\frac{\partial y^k}{\partial w^i}\frac{\partial y^l}{\partial w^j} + \phi_{,kl}\frac{\partial y^k}{\partial w^i}\frac{\partial y^l}{\partial w^j} - \phi_{,k}\frac{\partial^2 y^k}{\partial w^i \partial w^j}$$

$$\begin{bmatrix}
1+\frac{\partial\psi^1}{\partial x^1}\frac{\partial\psi^1}{\partial x^1} & \frac{\partial\psi^1}{\partial x^1}\frac{\partial\psi^1}{\partial x^2} & 0 & 0 & \frac{\partial\psi^1}{\partial x^1}\frac{\partial\psi^1}{\partial z^1} & \frac{\partial\psi^1}{\partial x^1}\frac{\partial\psi^1}{\partial z^2} & 0 & \frac{\partial\psi^1}{\partial x^1}\frac{\partial\psi^1}{\partial s^2} \\
 & 1+\frac{\partial\psi^1}{\partial x^2}\frac{\partial\psi^1}{\partial x^2} & 0 & 0 & \frac{\partial\psi^1}{\partial x^2}\frac{\partial\psi^1}{\partial z^1} & \frac{\partial\psi^1}{\partial x^2}\frac{\partial\psi^1}{\partial z^2} & 0 & \frac{\partial\psi^1}{\partial x^2}\frac{\partial\psi^1}{\partial s^2} \\
 & & 1+\frac{\partial\psi^2}{\partial x^3}\frac{\partial\psi^2}{\partial x^3} & \frac{\partial\psi^2}{\partial x^3}\frac{\partial\psi^2}{\partial x^4} & \frac{\partial\psi^2}{\partial x^3}\frac{\partial\psi^2}{\partial z^1} & \frac{\partial\psi^2}{\partial x^3}\frac{\partial\psi^2}{\partial z^2} & \frac{\partial\psi^2}{\partial x^3}\frac{\partial\psi^2}{\partial s^1} & 0 \\
 & & & 1+\frac{\partial\psi^2}{\partial x^4}\frac{\partial\psi^2}{\partial x^4} & \frac{\partial\psi^2}{\partial x^4}\frac{\partial\psi^2}{\partial z^1} & \frac{\partial\psi^2}{\partial x^4}\frac{\partial\psi^2}{\partial z^2} & \frac{\partial\psi^2}{\partial x^4}\frac{\partial\psi^2}{\partial s^1} & 0 \\
 & & & & 1+\frac{\partial\psi^k}{\partial z^1}\frac{\partial\psi^k}{\partial z^1} & \frac{\partial\psi^k}{\partial z^1}\frac{\partial\psi^k}{\partial z^2} & \frac{\partial\psi^2}{\partial z^1}\frac{\partial\psi^2}{\partial s^1} & \frac{\partial\psi^1}{\partial z^1}\frac{\partial\psi^1}{\partial s^2} \\
 & & & & & 1+\frac{\partial\psi^k}{\partial z^2}\frac{\partial\psi^k}{\partial z^2} & \frac{\partial\psi^2}{\partial z^2}\frac{\partial\psi^2}{\partial s^1} & \frac{\partial\psi^1}{\partial z^2}\frac{\partial\psi^1}{\partial s^2} \\
 & & & & & & 1+\frac{\partial\psi^2}{\partial s^1}\frac{\partial\psi^2}{\partial s^1} & 0 \\
 & \bullet & & & & & & 1+\frac{\partial\psi^1}{\partial s^2}\frac{\partial\psi^1}{\partial s^2}
\end{bmatrix}$$

Conclusions

- Foundations have been laid
- Early results have shown promise
- Areas of further inquiry have been identified



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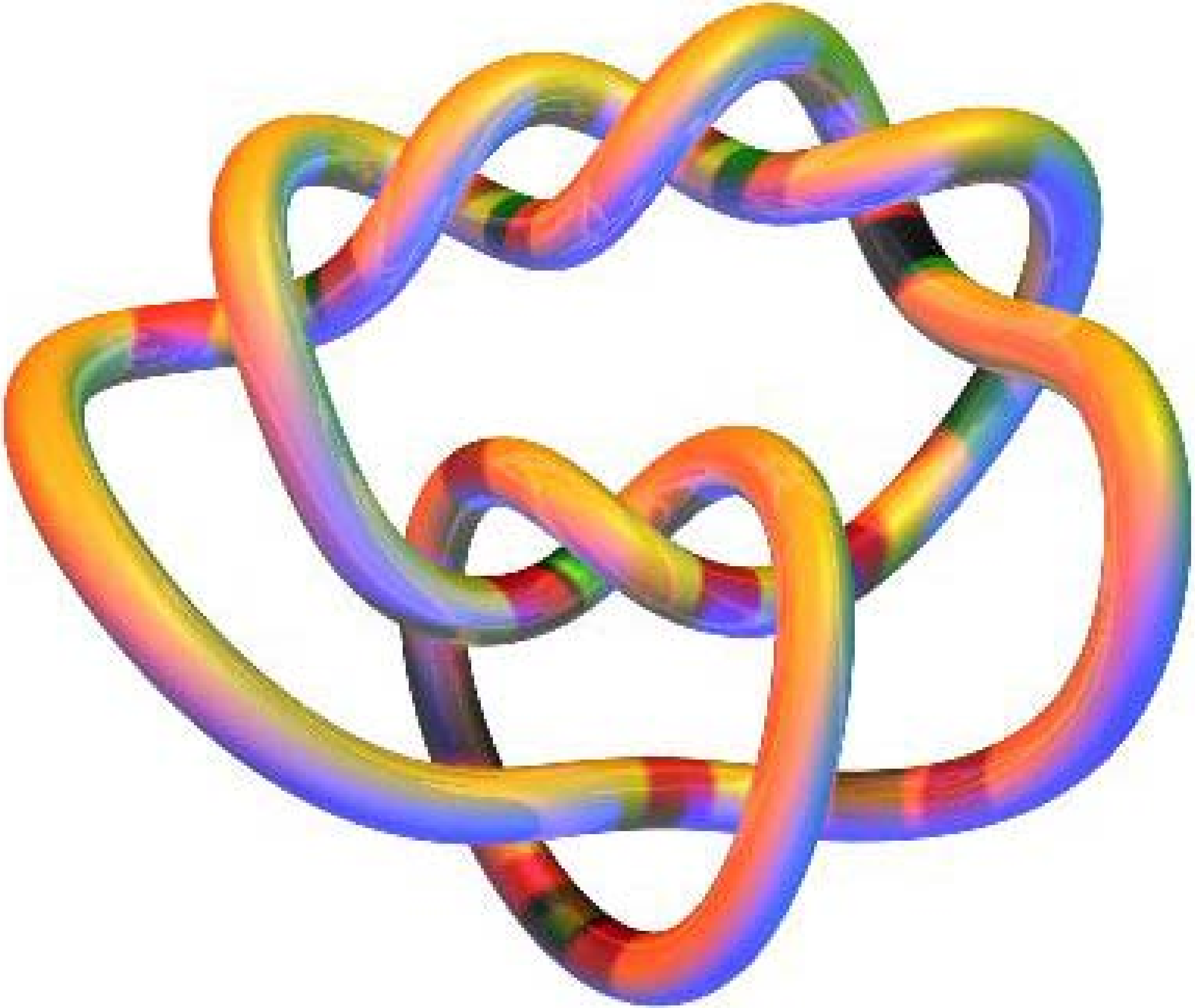
Questions?



Teichner, P.: pretzelhalf.jpeg. URL
http://math.berkeley.edu/~teichner/Bottoms.html