A new method for the linear stability analysis of turbulent flows

Vikrant Gupta, Matthew P. Juniper

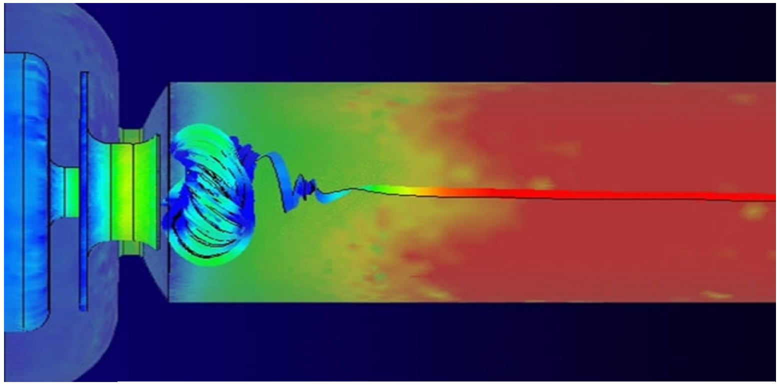
Department of Engineering

In combustion chambers, hydrodynamic instabilities can on one hand improve mixing between fuel and air, while on the other hand they can also lead to unwanted and harmful oscillations.

A method for hydrodynamic stability analysis of turbulent flows in industrial geometry that can:

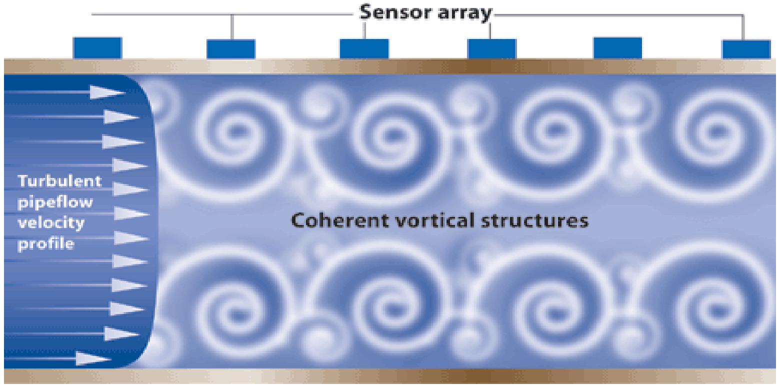
- identify the coherent structures in the flow.

- quickly identify the most influential regions of the flow.



http://www-edc.eng.cam.ac.uk/projects/combustor/

The modes with the largest transient growth or positive eigenvalues are related to the coherent structures observed in the real flows.



http://www.isa.org/Images/InTech/2004/January/20040115.gif

Linear stability analysis is performed for small perturbations around a base state, i.e. around a base flow.

General equation for a steady base flow in an incompressible laminar case:

1 L. $$\boxed{\overrightarrow{U}.\overrightarrow{\nabla}\overrightarrow{U} = -\frac{1}{\rho}\overrightarrow{\nabla}P + \nu\nabla^2\overrightarrow{U}.}$$

Consider small perturbations over the base flow:

2 L. $$\boxed{\overrightarrow{U}_{total} = \overrightarrow{U} + \overrightarrow{u}}$$ and $$\boxed{P_{total} = P + p}$$

The linear stability equation can be then written as:

3 L. $$\boxed{\frac{\partial\overrightarrow{u}}{\partial t} + \overrightarrow{u}.\overrightarrow{\nabla}\overrightarrow{U} + \overrightarrow{U}.\overrightarrow{\nabla}\overrightarrow{u} = -\frac{1}{\rho}\overrightarrow{\nabla}p + \nu\nabla^2\overrightarrow{u}.}$$

Turbulent flows are characterized by stochastic perturbations. The entity closest to the base flow is the turbulent mean flow, i.e. an ensemble average.

General equation for a steady mean flow in an incompressible turbulent case:

1 T. $$\boxed{\overrightarrow{\overline{U}}.\overrightarrow{\nabla}\overrightarrow{\overline{U}} = -\frac{1}{\rho}\overrightarrow{\nabla}\overline{P} + \nu\nabla^2\overrightarrow{\overline{U}} - \overrightarrow{\nabla}.\left[\overline{u_i'u_j'}\right].}$$

Consider small perturbations over the turbulent mean flow:

2 T. $$\boxed{\overrightarrow{U}_{total} = \overrightarrow{\overline{U}} + \overrightarrow{u}}$$ and $$\boxed{P_{total} = \overline{P} + p}$$

Are we allowed to take the perturbations over the turbulent mean flow?

It is not mathematically rigorous but still practically useful to identify the coherent structures in a turbulent flow.

Turbulent flows are characterized by stochastic perturbations. The entity closest to the base flow is the turbulent mean flow, i.e. an ensemble average.

General equation for a steady mean flow in an incompressible turbulent case:

1 T. $$\boxed{\overline{\overrightarrow{U}}.\overrightarrow{\nabla}\overline{\overrightarrow{U}} = -\frac{1}{\rho}\overrightarrow{\nabla}\overline{P} + \nu\nabla^2\overline{\overrightarrow{U}} - \overrightarrow{\nabla}.\left[\overline{u'_i u'_j}\right].}$$

Consider small perturbations over the turbulent mean flow:

2 T. $$\boxed{\overrightarrow{U}_{total} = \overline{\overrightarrow{U}} + \overrightarrow{u}}$$ and $$\boxed{P_{total} = \overline{P} + p}$$

Are we allowed to take the perturbations over the turbulent mean flow?

It is not mathematically rigorous but still practically useful to identify the coherent structures in a turbulent flow.

Challenge: What happens to the Reynolds stress terms?

1 T.

$$\boxed{\overrightarrow{\overline{U}}.\overrightarrow{\nabla}\overrightarrow{\overline{U}} = -\frac{1}{\rho}\overrightarrow{\nabla}\overline{P} + \nu\nabla^2\overrightarrow{\overline{U}} - \overrightarrow{\nabla}.\left[\overline{u_i'u_j'}\right].}$$

Physically : How does the background turbulence interact with the large scale perturbations?

Mathematically : How does the Reynolds stresses vary with the perturbations in the turbulent mean flow?

Literature review.

Simply ignore any changes in the Reynolds stresses.

3 T.
(same as 3 L.)

$$\boxed{\frac{\partial \overrightarrow{u}}{\partial t} + \overrightarrow{u}.\overrightarrow{\nabla}\overrightarrow{\overrightarrow{U}} + \overrightarrow{\overrightarrow{U}}.\overrightarrow{\nabla}\overrightarrow{u} = -\frac{1}{\rho}\overrightarrow{\nabla}p + \nu\nabla^2\overrightarrow{u}.}$$

Reynolds & Hussain (1972, JFM – 54) – classical linear stability analysis on a turbulent channel flow – found the mean flow is linearly stable.

Butler & Farrell (1992, Phys. Fluids A5) – non-modal stability analysis on a turbulent channel flow – related the modes with the largest transient growth to the coherent structures observed in the experiments.

Literature review (Eddy viscosity model).

The analysis is greatly improved by using the eddy viscosity.

$$\boxed{\nu_t = \nu_T + \nu}$$

[del-Alamo & Jimenez (2005, JFM – 559), Pujals *et al*. (2009, Phys. Fluids – 21)]

3 T. $$\boxed{\frac{\partial \overrightarrow{u}}{\partial t} + \overrightarrow{u}.\overrightarrow{\nabla}\overrightarrow{\overrightarrow{U}} + \overrightarrow{\overrightarrow{U}}.\overrightarrow{\nabla}\,\overrightarrow{u} = -\frac{1}{\rho}\overrightarrow{\nabla}p + \nabla.\left[\nu_t\left(\overrightarrow{\nabla}\,\overrightarrow{u} + \overrightarrow{\nabla}\,\overrightarrow{u}^T\right)\right].}$$

What goes on in this stability equation?

Eddy viscosity model.

Reynolds stress terms

$$\overline{u_i'u_j'} = -\nu_T \left[\frac{\partial \overline{U_i}}{\partial x_j} + \frac{\overline{U_j}}{\partial x_i}\right] + \frac{2}{3} K \delta_{ij}$$

$$\overline{u_i'u_j'}_{pert} = -\nu_T \left[\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\overline{u_j}}{\partial x_i}\right]$$

3 T.

$$\frac{\partial \overrightarrow{u}}{\partial t} + \overrightarrow{u}.\overrightarrow{\nabla}\overrightarrow{\overline{U}} + \overrightarrow{\overline{U}}.\overrightarrow{\nabla}\overrightarrow{u} = -\frac{1}{\rho}\overrightarrow{\nabla}p + \nabla.\left[\nu_t\left(\overrightarrow{\nabla}\overrightarrow{u} + \overrightarrow{\nabla}\overrightarrow{u}^T\right)\right]$$

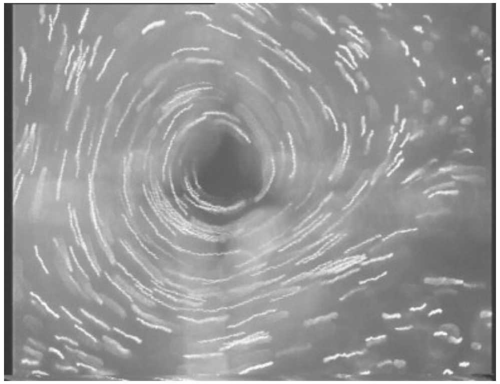
Are eddy viscosity models good enough?

Limitations of stability analysis methods based on eddy viscosity model.

Eddy viscosity models are first order closure models for turbulence.

- Gives good estimate for only one component of the Reynolds stress.

- Insensitive to three-dimensional flows.



Implementation of a better model is expected to improve the stability analysis and can be applicable for a wider range of flows.

- Which model?

A second order closure model: Explicit algebraic Reynolds stress models (EARSMs).

Reynolds stress models (RSMs).

- Differential equations $\Rightarrow$ difficult to implement in a stability analysis.

- Computationally expensive.

We have chosen explicit algebraic Reynolds stress models.

- Slightly less accurate than RSMs but very convenient.

- Reynolds stresses can be written in terms of mean flow quantities by the algebraic relations.

Explicit algebraic Reynolds stress models (EARSMs).

$$a_{ij} = \frac{\overline{u_i u_j}}{K} - \frac{2}{3}\delta_{ij}$$

$$S_{ij} = \frac{\tau}{2}\left(U_{i,j} + U_{j,i}\right) \qquad \Omega_{ij} = \frac{\tau}{2}\left(U_{i,j} - U_{j,i}\right)$$

The equations look big but the implementation in the linear stability analysis is, in principle, rather direct.

Explicit algebraic Reynolds stress models (EARSMs).

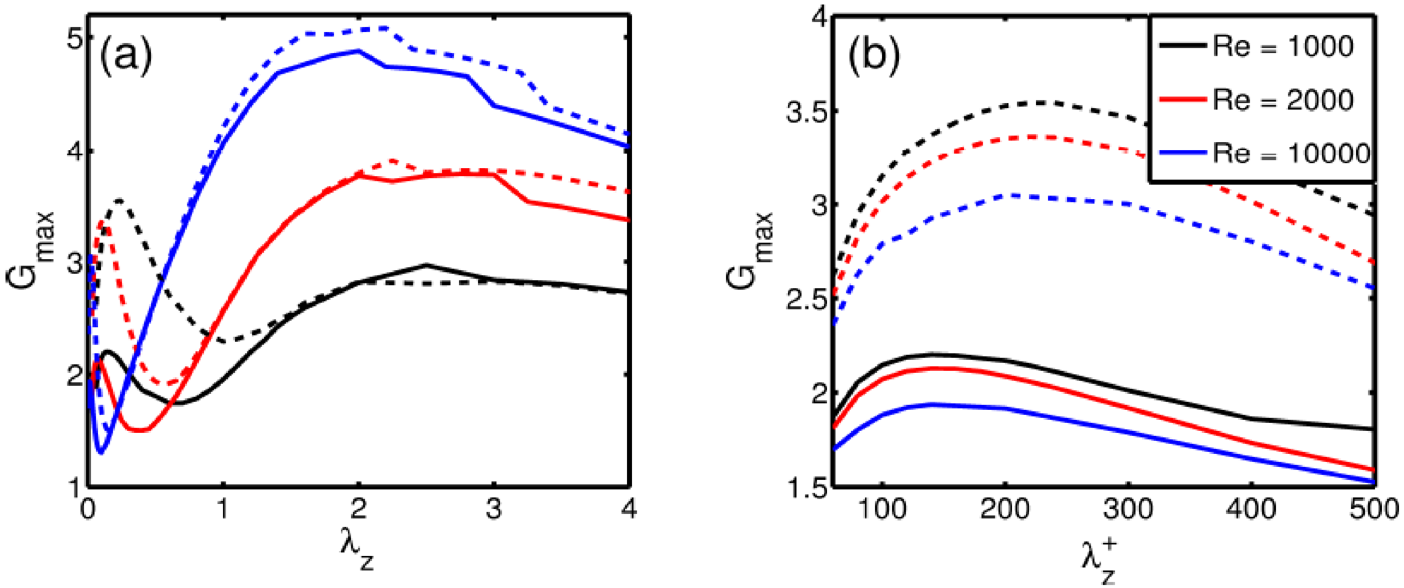
$$a_{ij} = \frac{\overline{u_i u_j}}{K} - \frac{2}{3}\delta_{ij}$$

$$S_{ij} = \frac{\tau}{2}\left(U_{i,j} + U_{j,i}\right) \qquad \Omega_{ij} = \frac{\tau}{2}\left(U_{i,j} - U_{j,i}\right)$$

The equations look big but the implementation in the linear stability analysis is, in principle, rather direct.

$$\begin{aligned}\mathbf{a} = \; & \beta_1 \mathbf{S} + \beta_2 \left(\mathbf{S}^2 - \tfrac{1}{3} II_S \mathbf{I}\right) + \beta_3 \left(\boldsymbol{\Omega}^2 - \tfrac{1}{3} II_\Omega \mathbf{I}\right) + \beta_4 \left(\mathbf{S}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{S}\right) + \beta_5 \left(\mathbf{S}^2\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{S}^2\right) \\ & + \beta_6 \left(\mathbf{S}\boldsymbol{\Omega}^2 + \boldsymbol{\Omega}^2\mathbf{S} - \tfrac{2}{3} IV \mathbf{I}\right) + \beta_7 \left(\mathbf{S}^2\boldsymbol{\Omega}^2 + \boldsymbol{\Omega}^2\mathbf{S}^2 - \tfrac{2}{3} V \mathbf{I}\right) + \beta_8 \left(\mathbf{S}\boldsymbol{\Omega}\mathbf{S}^2 - \mathbf{S}^2\boldsymbol{\Omega}\mathbf{S}\right) \\ & + \beta_9 \left(\boldsymbol{\Omega}\mathbf{S}\boldsymbol{\Omega}^2 - \boldsymbol{\Omega}^2\mathbf{S}\boldsymbol{\Omega}\right) + \beta_{10} \left(\boldsymbol{\Omega}\mathbf{S}^2\boldsymbol{\Omega}^2 - \boldsymbol{\Omega}^2\mathbf{S}^2\boldsymbol{\Omega}\right),\end{aligned}$$

Results for a channel flow: near-wall region is still a challenge.



Two peaks in the transient growth are identified:
1. At the spanwise wavelength around 2 (in the outer units).
2. At the spanwise wavelength around 100 (in the inner units, i.e. wall units).

They match with the observations from the experiments and DNS.

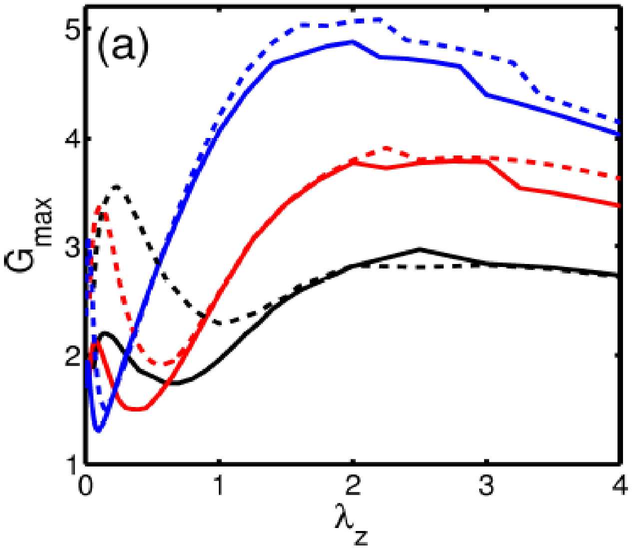
Turbulence models often fail in the near-wall region?

Summary and future work

Linear stability analysis is useful in identifying the coherent structures in flows.

For complex flows better turbulence models need to be implemented.

We choose explicit algebraic Reynolds stress model and currently testing the stability analysis based on it for a few simpler flows.



The ultimate goal is to apply the technique for an industrial flow and evaluate the influence of using different turbulence models in case of combustion chambers.